A Trace Inequality for *M*-matrices and the Symmetrizability of a Real Matrix by a Positive Diagonal Matrix

Miroslav Fiedler
Czechoslovak Academy of Sciences
Institute of Mathematics
Žitná 25
Prague, Czechoslovakia
and
Department of Mathematics and Statistics
University of South Carolina
Columbia, South Carolina 29208

Charles R. Johnson
Department of Mathematical Sciences
Clemson University
Clemson, South Carolina 29631

and

Thomas L. Markham and Michael Neumann
Department of Mathematics and Statistics
University of South Carolina
Columbia, South Carolina 29208

Dedicated to Helmut Wielandt on his 75th birthday

Submitted by Hans Schneider

ABSTRACT

The question of whether a real matrix is symmetrizable via multiplication by a diagonal matrix with positive diagonal entries is reduced to the corresponding question for M-matrices and related to Hadamard products. In the process, for a nonsingular M-matrix A, it is shown that $\operatorname{tr}(A^{-1}A^T)\leqslant n$, with equality if and only if A is symmetric, and that the minimum eigenvalue of $A^{-1}\circ A$ is $\leqslant 1$ with equality in the irreducible case if and only if A is positive diagonally symmetrizable.

INTRODUCTION

Motivated by the issue of which real n-by-n matrices may be symmetrized by a positive diagonal multiplication, certain questions about M-matrices are

52 Vanderbilt Ave., New York, NY 10017 0024-3795/85/$3.30

raised and answered. It is shown that if A is an n-by-n nonsingular M-matrix, then $\mathrm{tr}(A^{-1}A^T) \leqslant n$, with equality holding if and only if A is symmetric. From this it follows for a nonsingular M-matrix A that the minimum (real) eigenvalue of the Hadamard product $A^{-1} \circ A$ (necessarily an M-matrix) is less than or equal to 1, with equality holding for irreducible A if and only if A is positive diagonally symmetrizable. This was originally conjectured in [6]. This suggests the minimum real eigenvalue of $A^{-1} \circ A$ as a measure of the symmetrizability of A and permits characterization of the positive diagonal symmetrizability of general real matrices in terms of Hadamard products and eigenvalues.

POSITIVE DIAGONAL SYMMETRIZABILITY

An n-by-n matrix A is said to be *diagonally symmetrizable* (DS) if there exists a nonsingular n-by-n diagonal matrix X such that

$$AX \text{ is symmetric.} \tag{1}$$

Since $X^{-1}A = X^{-1}(AX)X^{-1}$ and $X^{-1/2}AX^{1/2} = X^{-1/2}(AX)X^{-1/2}$, in which $X^{1/2}$ is a particular square root of X with inverse $X^{-1/2}$, the symmetry of AX is equivalent to that of $X^{-1}A$ or of $X^{-1/2}AX^{1/2}$. Thus, we could just as well have defined diagonal symmetrizability via left diagonal multiplication or via diagonal similarity. We note that the symmetrizability of A is independent of the diagonal entries of A. Thus, without loss of generality, we may assume that A is nonsingular and may prescribe the diagonal entries of A as convenient.

Our primary interest is in real matrices A for which (1) holds for X with positive diagonal entries. In this case, we call A *positive diagonally symmetrizable* (PDS). We note that as a practical computational matter, the determination of whether a given real n-by-n matrix is PDS is not a difficult problem, since the solvability in $x_i > 0$, $i = 1,\dots,n$, of the $n(n-1)/2$ equations $a_{ij}x_j = a_{ji}x_i$ may be determined in a straightforward way. Also, the diagonal symmetrizability of A has been constructed previously [2] from the point of view of the cycle-product structure of A. Our focus, however, is upon the relationship between positive diagonal symmetrizability on the one hand, and M-matrices and Hadamard products on the other. In a later section, we show that the question of whether a general real matrix is PDS may be reduced to that question for an M-matrix and give a characterization in the M-matrix case.

NOTATION AND BACKGROUND

Denote the Perron eigenvalue of an n-by-n componentwise nonnegative matrix P by $p(P)$. An M-matrix is simply an n-by-n matrix A of the form $A = \alpha I - P$ in which P is nonegative and $\alpha \geqslant p(P)$. If $\alpha = p(P)$, then A is a singular M-matrix, and if $\alpha > p(P)$, then A is a nonsingular M-matrix. Denote the minimum real eigenvalue of an M-matrix A by $q(A)$. Then $q(A) = \alpha - p(P)$, and $q(A)$ is also the minimum of the real parts of the eigenvalues of A. See [3], for example, for further discussion of M-matrices.

The Hadamard (entrywise) product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same dimensions is denoted by $\circ$ and defined by

$$A \circ B \equiv \left(a_{ij} b_{ij}\right). \tag{2}$$

In [5] the following is proven.

THEOREM. *Let P be an irreducible n-by-n nonnegative matrix, with $p(P) = p$, and let u and v be positive vectors such that $Pu = pu$ and $P^T v = pv$. Then*

(a) $\displaystyle \max_{z>0} \min_{\substack{x>0,\, y>0 \\ x \circ y = z}} \frac{y^T P x}{y^T x} = p$;

(b) $y^T P x \geqslant v^T P u$ *whenever* $x > 0$ *and* $y > 0$ *satisfy* $x \circ y = u \circ v$, *and equality is attained if and only if x and u are linearly dependent; and*

(c) *in particular,* $u^T P v \geqslant v^T P u$, *with equality if and only if u and v are linearly dependent.*

Proof. Let $P = (p_{ij})$ be of order n, $u = (u_1, \dots, u_n)^T$, and $v = (v_1, \dots, v_n)^T$. Define

$$U = \operatorname{diag}\{u_i, \dots, u_n\}, \qquad V = \operatorname{diag}\{v_1, \dots, v_n\}, \tag{3}$$

$$\Delta = \operatorname{diag}\{pu_1 v_1, \dots, pu_n v_n\}, \qquad \sigma = \left(1 + \max_k \left[(p - p_{kk}) u_k v_k\right]\right)^{-1}. \tag{4}$$

It is easily checked that the matrix D defined by

$$D = \sigma(VPU - \Delta) + I \tag{5}$$

is doubly stochastic. By Birkhoff's theorem, D has the form

$$D = \sum_k \lambda_k P_k, \tag{6}$$

where the P_k are permutation matrices and the λ_k are nonnegative numbers with sum 1.

Let $x > 0$, $y > 0$ satisfy $x \circ y = u \circ v$. Define the vector z by $x = u \circ z$, the vector z^{-1} by $y = v \circ z^{-1}$. Since $x \circ y = u \circ v$, we have $z \circ z^{-1} = e$, and also $z > 0$, $z^{-1} > 0$.

Let us show that

$$(z^{-1})^T Dz \geqslant e^T De. \tag{7}$$

This is an immediate consequence of (6) and the fact that for any permutation matrix P_k,

$$(z^{-1})^T P_k z \geqslant n = e^T P_k e, \tag{8}$$

the left-hand side being a consequence of the inequality

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_s}{x_1} \geqslant s$$

following from the arithmetic–geometric-mean inequality for positive numbers. Clearly, equality in (8) is attained if and only if the coordinates of z are equal for indices from every cycle of the permutation corresponding to P_k. Since D is irreducible, every pair of indices can be joined by a sequence of pairs occurring in some cycle of some permutation P_k which is present in (6) with a positive λ_k. Therefore, equality in (7) is attained if and only if all coordinates of z coincide.

Now, substituting from (5) into (7), we obtain

$$\sigma\big((z^{-1})^T VPUz - (z^{-1})^T \Delta z\big) + (z^{-1})^T z \geqslant \sigma(e^T VPUe - e^T \Delta e) + e^T e.$$

Since Δ is diagonal and $z^{-1} \circ z = e$, it follows that

$$(z^{-1})^T VPUz \geqslant e^T VPUe,$$

which is easily seen to be

$$y^T P x \geqslant v^T P u,$$

i.e. (b). By the previous statement about equality in (b), equality is attained as asserted.

(c) being a trivial consequence of (b), it remains to prove (a).

For $z > 0$, define

$$\phi(z) = \inf\left\{ \frac{y^T P x}{y^T x};\ x > 0,\ y > 0,\ x \circ y = z \right\}.$$

To show that $\phi(z)$ is, in fact, a minimum, denote by Ω the set of all positive vectors $x = (x_i)$ satisfying $\sum x_i = 1$. Let Ω_z denote the set of all $x \in \Omega$ for which, if y is defined by $x \circ y = z$, we have $y^T P x / y^T x \leqslant p(P)$. Since $u \in \Omega_z$, Ω_z is not void. We shall show that Ω_z is closed. Let $\{x^{(k)}\}$ be a convergent sequence, $x^{(k)} \in \Omega_z$, $k = 1, 2, \dots$. Suppose that $x = \lim x^{(k)}$ does not belong to Ω_z. Clearly at least one coordinate of $x = (x_i)$ is zero; let $M = \{i;\ x_i = 0\}$. If $y^{(k)} = (y_i^{(k)})$ is defined by $x^{(k)} \circ y^{(k)} = z$, we have

$$y^{T(k)} P x^{(k)} \leqslant p(P) \sum z_i, \qquad \text{where} \qquad z = (z_i).$$

On the other hand,

$$y^{T(k)} P x^{(k)} \geqslant \sum_{i \in M} \sum_{j \notin M} P_{ij} x_j^{(k)} y_i^{(k)}.$$

Since

$$\sum_{j \notin M} P_{ij} x_j^{(k)} \to \sum_{j \notin M} P_{ij} x_j$$

and the last numbers are not all equal to zero for $i \in M$ because of the irreducibility of P, at least one of the sequences $\{y_i^{(k)}\}$, $i \in M$, has to be bounded from above, a contradiction with $x^{(k)} \circ y^{(k)} = z$ and $x^{(k)} \to 0$ for $i \in M$. Therefore, Ω_z is closed, so that $\phi(z)$ is the minimum.

Now, let

$$M = \sup\{\phi(z);\ z > 0\}.$$

By the previous result, $M \leqslant p(P)$. On the other hand, (b) implies that

$$\phi(u \circ v) = p(P),$$

i.e.

$$M \geqslant p(P).$$

Thus $M = p(P)$ and the supremum is a maximum. ■

The following results from some observations in [4] and was noted explicitly in [6].

THEOREM. *If A and B are n-by-n nonsingular M-matrices, then $A^{-1} \circ B$ is again an M-matrix. In particular,*

$$A^{-1} \circ A \text{ is an M-matrix}. \tag{9}$$

We denote the column vector all of whose entries are 1 by e throughout. The following elementary fact will be useful. Let $C = (c_{ij})$ be an n-by-n matrix. If we consider $\det C$ as a function of all entries of C, then the (i, j)th entry of the adjoint matrix $\operatorname{adj} C$ may be written as

$$(\operatorname{adj} C)_{ij} = \frac{\partial}{\partial c_{ji}} \det C. \tag{10}$$

A TRACE INEQUALITY FOR M-MATRICES

Although our principal results will be given later, they will be consequences of the following observation which is of interest in its own right.

THEOREM 1. *Let $A = (a_{ij})$ be an n-by-n (possibly singular) M-matrix. Then*

$$\sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})\left[(\operatorname{adj} A)_{ji} - (\operatorname{adj} A)_{ij}\right] \geqslant 0. \tag{11}$$

If A is nonsingular, equality occurs if and only if A is symmetric.

Proof. We first prove (11) in the case in which A is singular. If A is irreducible, then $Au = 0$ for some $u > 0$, $A^T v = 0$ for some $v > 0$, and

$$\operatorname{adj} A = cuv^T \qquad \text{for some scalar } c > 0.$$

The left-hand side of (11) may then be written as

$$c \sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})(u_j v_i - u_i v_j) = c(v^T A u - u^T A v)$$

Since A is of the form $A = pI - P$ with $P \geqslant 0$, $p = p(P)$ and with $Pu = pu$, $P^T v = pv$, it follows from (4) that

$$v^T A u \geqslant u^T A v, \tag{12}$$

and (11) holds in this case.

If A is (singular and) reducible, we may assume that A has already the block triangular form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{2r} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{rr} \end{pmatrix} \tag{13}$$

(in which the blocks A_{uu} are irreducible) because the left-hand side of (11) is unchanged by permutation similarity of A. Let A_{tt} lie in the rows and columns corresponding to the index set N_t. Since $\operatorname{adj} A$ is also of the form (13), the left-hand side of (11) may be written as

$$\sum_{t=1}^{r} \sum_{\substack{i<j \\ i,\, j \in N_t}} (a_{ij} - a_{ji})\left[(\operatorname{adj} A)_{ji} - (\operatorname{adj} A)_{ij}\right]$$

$$+ \sum_{\substack{s,\, t = 1 \\ s < t}} \sum_{\substack{i \in N_s \\ j \in N_t}} a_{ij}\left[-(\operatorname{adj} A)_{ij}\right].$$

Since $\operatorname{adj} A \geqslant 0$ and since the off-diagonal entries of A are nonpositive, the second summand is nonnegative. Since A is singular, at least one of the matrices A_{uu}, say A_{ss}, is singular. Therefore, each diagonal block of $\operatorname{adj} A$ is zero, except perhaps that corresponding to N_s. However, diagonal block s of $\operatorname{adj} A$ is nonnegative multiple of $\operatorname{adj}(A_{ss})$, and, since A_{ss} is irreducible, the nonnegativity of the first sum follows from the previously studied case. Thus (11) holds for arbitrary singular M-matrices.

The remainder of the proof will consist of showing that the inequality (11) also holds in the case A is a nonsingular M-matrix, followed by considering the case of equality in (11). Both of these parts of the proof are demonstrated by induction on n.

We first show that (11) holds when A is nonsingular. For $n = 1$, the assertion is immediate. For $n = 2$,

$$\operatorname{adj} A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$

so that (11) is equivalent to

$$(a_{12} - a_{21})(-a_{21} + a_{12}) \geqslant 0,$$

which is evidently the case. Thus, let $n > 2$ and assume that (11) holds for all nonsingular M-matrices of size $n-1$.

For $x \geqslant 0$, denote $A(x)$ the matrix

$$A(x) = A + (x - q)I,$$

where $q = q(A)$, the least real eigenvalue of A. Since $q > 0$, we have $A = A(q)$, and moreover $A(0)$ is a singular M-matrix. Next, for $x \geqslant 0$, define the function

$$f(x) = \sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})\left[(\operatorname{adj} A(x))_{ji} - (\operatorname{adj} A(x))_{ij}\right], \tag{14}$$

the left-hand side of (11) for $A(x)$. We wish to show that $f(q) \geqslant 0$. In the first part of the proof we have seen that

$$f(0) \geqslant 0.$$

In fact, we shall show that $f(x) \geqslant 0$ for $x \geqslant 0$ by demonstrating that the derivative

$$f'(x) \geqslant 0 \qquad \text{for} \quad x \geqslant 0. \tag{15}$$

Let $A_m(x)$, $m = 1,\dots,n$, denote the principal submatrix of $A(x)$ resulting from deletion of row m and column m from $A(x)$. We claim that

$$f'(x) = \sum_{m=1}^{n} \sum_{\substack{1 \leqslant i < j \leqslant n \\ i \neq m \neq j}} (a_{ij} - a_{ji})\left[(\operatorname{adj} A_m(x))_{ji} - (\operatorname{adj} A_m(x))_{ij}\right]. \tag{16}$$

Indeed, considering, for a moment, $A(x)$ as a function of x as well as of all the entries a_{ij}, $i \neq j$, and using (10), we may write

$$f(x) = \sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})\left(\frac{\partial}{\partial a_{ij}} \det A(x) - \frac{\partial}{\partial a_{ji}} \det A(x)\right).$$

Therefore,

$$\begin{aligned} f'(x) &= \sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})\left(\frac{\partial^2}{\partial x\, \partial a_{ij}} \det A(x) - \frac{\partial^2}{\partial x\, \partial a_{ji}} \det A(x)\right) \\ &= \sum_{1 \leqslant i < j \leqslant n} (a_{ij} - a_{ji})\left(\frac{\partial^2}{\partial a_{ij}\, \partial x} \det A(x) - \frac{\partial^2}{\partial a_{ji}\, \partial x} \det A(x)\right). \end{aligned} \tag{17}$$

Since $(\partial / \partial x)\det A(x) = \sum_{m=1}^{n} \det A_m(x)$, we obtain that

$$f'(x) = \sum_{m=1}^{n} \sum_{\substack{1 \leqslant i < j \leqslant n \\ i \neq m \neq j}} (a_{ij} - a_{ji})\left(\frac{\partial}{\partial a_{ij}} \det A_m(x) - \frac{\partial}{\partial a_{ji}} \det A_m(x)\right), \tag{18}$$

from which (16) follows upon another application of (10). (We note that $A_m(x)$ does not contain a_{im} or a_{mi} for $i \neq m$.)

Since $A_m(x)$ is an M-matrix of order $n-1$, which is nonsingular when $x > 0$, the summands in (16) are nonnegative by the induction hypothesis. This completes the proof of the inequality (11).

We finally turn to prove our assertion that equality occurs in (11) if and only if A is symmetric. The sufficiency of the symmetry condition is clear, so that we need only verify the necessity of this condition. Again, this statement is a straightforward calculation for $n = 1$ and $n = 2$. Assume the assertion for M-matrices of order $n-1$ and let A be an n-by-n M-matrix. For $x > 0$, we can have $f(x)=0$ only if $f'(y)=0$ for all y, $0<y<x$. By the induction hypothesis again, $A_m(y)$, which is then a nonsingular M-matrix of order $n-1$, has to be symmetric for $m=1,\dots,n$. It follows that A is symmetric, which completes the proof. ■

It is useful to observe that, for an arbitrary n-by-n nonsingular matrix A,

(i) the ith row sum of $A^{-1}\circ A$ is the ith diagonal entry of $A^{-1}A^t$, and
(ii) the ith row sum of $A^{-1}\circ A^T$ is the ith diagonal entry of $A^{-1}A$, i.e. 1.

For nonsingular M-matrices A, the inequality (11) of Theorem 1 is equivalent to

$$e^T\left[(A-A^T)\circ(A^{-1T}-A^{-1})\right]e\geqslant 0, \tag{19}$$

because $\det A>0$ and because $(A-A^T)\circ(A^{-1T}-A^{-1})$ is a symmetric matrix with zero diagonal entries. Since the Hadamard product is commutative and commutes with transposition, (19) is equivalent to

$$e^T(A^{-1}\circ A)e\leqslant e^T(A^{-1}\circ A^T)e. \tag{20}$$

Because of (18), $e^T(A^{-1}\circ A^T)e=n$ if A is n-by-n. We conclude, by applying (17) to the left-hand side of (20), our principal result.

THEOREM 2. *If A is an n-by-n nonsingular M-matrix, then*

$$\operatorname{tr}(A^{-1}A^T)\leqslant n, \tag{21}$$

with equality of and only if A is symmetric.

Note that Theorem 2 implies Theorem 1 also.

Since congruences of A transform to similarities of the form $A^{-1}A^T$, it follows immediately from Theorem 2 that:

COROLLARY 3. *Let A be an n-by-n matrix. If there exists an n-by-n matrix W such that*

$$B=W^TAW$$

is a nonsingular M-matrix, then

$$\operatorname{tr}(A^{-1}A^T)\leqslant n.$$

Equality occurs if and only if A is symmetric.

Proof. For B to be a (nonsingular) M-matrix, W and A must be nonsingular, and the assertion follows from the calculation that $B^{-1}B^T$ is similar via W to $A^{-1}A^T$ and that A is symmetric if and only if B is symmetric.

For an n-by-n matrix B, the Hermitian part $H(B)$ is defined by

$$H(B) = \tfrac{1}{2}(B + B^*).$$

If a real matrix A has nonpositive off-diagonal entries, it is well known that the positive definiteness of $H(A)$ is sufficient for A to be an M-matrix. For any M-matrix A, there is a positive diagonal matrix D such that $H(AD)$ is positive definite. It is worth noting that there is a quite different and much simpler proof of (21) under the alternative assumption that $H(A)$ is positive definite. In [1], it is shown that if $H(A)$ is positive definite, than $A^{-1}A^*$ is similar to a unitary matrix. Thus $\mathrm{tr}(A^{-1}A^*) \leqslant n$, and, since the only unitary matrix with trace n is the identity, equality holds if and only if A is Hermitian. Analogously, if $H(AD)$ is positive definite for the positive diagonal matrix D (and such a D exists for every nonsingular M-matrix, as noted above), then $\mathrm{tr}(D^{-1}A^{-1}DA^T) \leqslant n$, with equality if and only if AD is symmetric. It would be interesting to know if $\max \mathrm{tr}(D^{-1}A^{-1}DA^T)$ occurs for a D such that $H(AD)$ is positive definite. If so, it would not only provide an alternative proof to Theorem 2, but also give an interesting characterization of a particular diagonal Lyapunov solution for A.

Besides the applications of the next section, there are a number of interesting consequences of Theorem 2. Among these are various manipulations of (19). Another is the following. Let P be an n-by-n nonnegative matrix with $p(P) < 1$. Then $I - P$ is an M-matrix, and $(I-P)^{-1}(I-P^T) = (I + P + P^2 + \cdots)(I - P^T) = I + (P - P^T) + P(P-P^T) + \cdots$. Application of (21) yields

$$\mathrm{tr}\big(P(P-P^T) + P^2(P-P^T) + \cdots\big) \leqslant 0$$

or

$$\mathrm{tr}(P^2 + P^3 + P^4 + \cdots) \leqslant \mathrm{tr}(PP^T + P^2P^T + P^3P^T + \cdots).$$

It is clear that $\mathrm{tr}(P^2) \leqslant \mathrm{tr}(PP^T)$, but $\mathrm{tr}(P^{k+1})$ is not $\leqslant \mathrm{tr}(P^kP^T)$ for $k \geqslant 2$, in general.

THE APPLICATION TO POSITIVE DIAGONAL SYMMETRIZABILITY

We first indicate that the question of whether a general real matrix is PDS may be reduced to the question of whether a nonsingular M-matrix is PDS. The latter question is then treated in a final theorem which follows from Theorem 2.

An n-by-n real matrix $A=(a_{ij})$ is said to be *strongly combinatorially symmetric* if $a_{ij}a_{ji}\geqslant 0$ for all $i,\ j\leqslant n$, and $a_{ij}a_{ji}=0$ only when $a_{ij}=a_{ji}=0$, that is, A and A^T have the same $+,\ -,0$ sign pattern. It is clear that strong combinatorial symmetry is a necessary condition for A to be PDS. Furthermore, under the assumption that this necessary condition holds, A is PDS if and only if the comparison matrix $M(A)=(m_{ij})$ defined by

$$m_{ij}\equiv\begin{cases}|a_{ii}|, & i=1,\dots,n,\\ -|a_{ij}|, & i\neq j,\end{cases}$$

is PDS. Given that the combinatorial condition for PDS is met, the matrix $M(A)$ simply isolates the remaining magnitudinal question. Since the diagonal of A is immaterial to the question of whether A is PDS, we may further assume that the diagonal of A is sufficiently large so that A is a nonsingular M-matrix. Should a strongly combinatorially symmetric matrix A be reducible, it would necessarily be completely reducible (permutation similar to a direct sum), and then its positive diagonal symmetrizability would be a question of that of each of its summands. In light of the above discussion, the PDS question for real matrices, for which strong combinatorial symmetry has been verified, reduces to the PDS question for irreducible M-matrices. Henceforth we discuss only M-matrices.

In [6] it was shown that if A is a nonsingular M-matrix which is PDS, then the minimum root $q(A^{-1}\circ A)$ of the M-matrix $A^{-1}\circ A$ satisfies

$$q(A^{-1}\circ A)=1. \tag{22}$$

We may now show the converse for irreducible A and a general inequality for $q(A^{-1}\circ A)$.

THEOREM 4. *Let A be an M-matrix. Then*

$$q(A^{-1}\circ A)\leqslant 1. \tag{23}$$

For irreducible A, equality occurs if and only if A is PDS.

Proof. For reducible A, the matrices A and A^{-1}, and therefore $A \circ A^{-1}$, have irreducible components determined by the same index sets. Since $q(A^{-1} \circ A)$ is attained for one of these, it suffices to consider only irreducible A to verify the general inequality (23). We may (because $A^{-1} \circ A$ would then be irreducible) therefore suppose that there is a positive eigenvector x corresponding to $q(A^{-1} \circ A)$. Let the diagonal matrix X be defined by $x = Xe$. Then $(A^{-1} \circ A)x = q(A^{-1} \circ A)x$ implies $[(AX)^{-1} \circ AX]e = q(A^{-1} \circ A)e$ and $e^T[(AX)^{-1} \circ AX]e = nq(A^{-1} \circ A)$. By (17), the latter means $\operatorname{tr}((AX)^{-1}(AX^T) = nq(A^{-1} \circ A)$. Since AX is an M-matrix, Theorem 2 implies that $nq(A^{-1} \circ A) \leqslant n$, or that (23) holds. If A is actually irreducible, Theorem 2 further implies that $q(A^{-1} \circ A) = 1$ if and only if AX is symmetric, that is, if and only if A is PDS. ∎

EXAMPLE. It is important to note that $q(A^{-1} \circ A) = 1$ does not imply that A is PDS for general (nonsingular) M-matrices. If

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

then

$$A^{-1} \circ A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and $q(A^{-1} \circ A) = 1$. Evidently A is not PDS, as it is not even strongly combinatorially symmetric. The problem, of course, is that $A^{-1} \circ A$ does not have a positive eigenvector corresponding to $q(A^{-1} \circ A)$. The existence of one could replace the irreducibility assumption in Theorem 4, and if $q(A^{-1} \circ A) = 1$, the associated (positive) eigenvector would give the diagonal entries of a diagonal symmetrizer. Note further that if A is reducible and $(A^{-1} \circ A)x = q(A^{-1} \circ A)x$, with x only nonnegative, then AX is symmetric for $x = Xe$, but X may be singular.

QUESTION. If A is a nonsingular n-by-n M-matrix, we know that

$$0 < q(A^{-1} \circ A) \leqslant 1.$$

The left-hand inequality is because $A^{-1} \circ A$ is a nonsingular M-matrix, and the right-hand inequality is (23). Is there a sharp lower bound larger than 0, perhaps dependent on n, for $q(A^{-1} \circ A)$? Because of Fischer's inequality for M-matrices, each diagonal entry of $A^{-1} \circ A$ is bounded below by 1, with the inequality strict in the irreducible case. We comment here that from this lower bound on the diagonal entries of $A \circ A^{-1}$ it follows that $\operatorname{tr}(A^{-1} \circ A) \geqslant n$. Hence $A^{-1} \circ A$ must have an eigenvalue whose real part is bounded below by unity. Thus

$$\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda;\ \det(\lambda I - A) = 0\} \geqslant 1.$$

It follows then that also the spectral radius of $A^{-1} \circ A$ is bounded below by 1.

ADDENDUM

Fiedler's theorem, which is proved at the beginning of this paper, is proved (in a different manner) in a preprint by B. C. Eaves, A. J. Hoffman, U. G. Rothblum, and H. Schneider, "Line-sum-symmetric scalings of square nonnegative matrices." This paper will appear in *Math. Programming*.

REFERENCES

1 C. R. DePrima and C. R. Johnson, The range of $A^{-1}A^*$ in $GL(n, C)$, *Linear Algebra Appl.* 9:209–222 (1974).
2 G. Engel and H. Schneider, Matrices diagonally similar to a symmetric matrix, *Linear Algebra Appl.* 29:131–138 (1980).
3 M. Fiedler and V. Pták, On matrices with nonpositive off-diagonal elements and positive principal minors, *Czechoslovak Math. J.* 42:382–400 (1962).
4 M. Fiedler and V. Pták, Diagonally dominant matrices, *Czechoslovak Math. J.* 47:420–433 (1967).
5 M. Fiedler, An alternative proof of Levinger's theorem, unpublished, 1970.
6 C. R. Johnson, A Hadamard product involving M-matrices, *Linear and Multilinear Algebra* 4:261–264 (1977).

Received 7 May 1984; revised 15 January 1985